

Effect of cooling rate on the solidification of a cylindrical channel section.

Guy J. Ruddock and A. J. M. Spencer

Department of Theoretical Mechanics

Nottingham University

Nottingham

NG7 2RD

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Abstract. This paper presents a model of solidification for a fibre-reinforced circular cylindrical channel section under plane strain. Cooling from the inside surface of a cylindrical section of the liquid phase causes a solidification front to move radially through the thickness until solidification is complete. Deformation which occurs in the liquid phase becomes "locked in" at solidification, thus affecting the subsequent behaviour of the material. The driving force behind this mechanism is the temperature distribution during solidification, which in turn depends on the cooling rate; the paper quantifies the effects of this on the shape and residual stresses in the final solid part. The model includes the so-called "spring-forward" result as a special case.

1. Introduction

Existing models of the sheet forming process for thermoplastic fibre-reinforced materials generally consider only one phase of the material. This phase may be viscoelastic or thermoelastic, depending on whether the focus of study is the flow near solidification, or the subsequent cooling and distortion of the solid component. However, Ruddock [1] recently offered a genuinely two-phase model of solidification for a cylindrical sector in which a surface separating the liquid from the forming solid gradually advances through the component. The solidification of a rectangular slab was investigated in Ruddock and Spencer [2].

Results from this model are both qualitatively and quantitatively different from an equivalent analysis which considers only the solid phase (as given, for example, in O'Neill, Rogers and Spencer [3], and Spencer, Watson and Rogers [4, 5]). The crucial feature of the two-phase analysis is that the solid region deforms as a result of the falling temperature, and that this deformation causes shear in the liquid which becomes locked in at solidification.

The present paper investigates the possibility of stress-free solutions during the solidification of a circular cylindrical sector of fibre-reinforced material. The material, which is assumed to be orthotropic, is initially in its liquid state, resting on a circular cylindrical mould, cf Figure 1. The inner surface is cooled, and as soon as solidification commences there, the deformation of the liquid is controlled by the temperature-induced deformation of the solid. In particular, the liquid is free to pull away from the mould, and it is assumed to be sufficiently viscous that it does not simply flow away due to gravity.

The model incorporates a volume change on solidification, and allows for different thermal behaviour in the two phases. The specific emphasis is on the relationship between the cooling rate and the change in curvature of the sector.

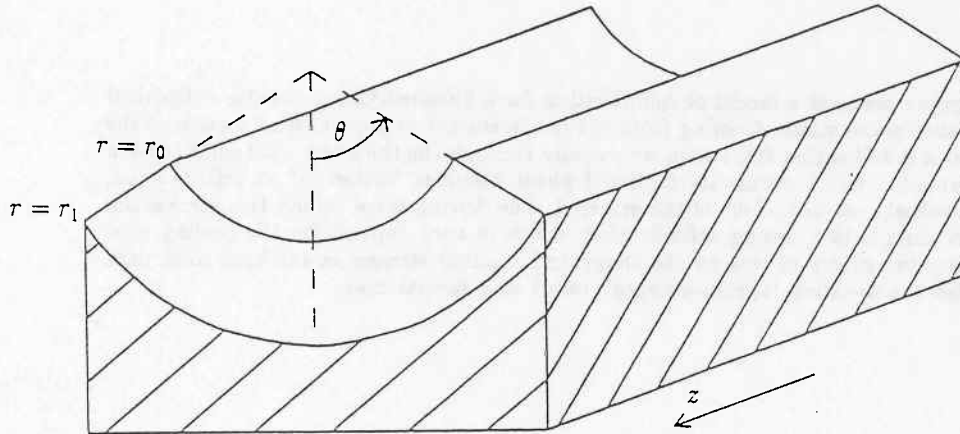


Fig. 1. Cylindrical channel section (the mould is hatched).

2. Formulation

It is assumed that at time $t = 0$ the material is in its liquid state at some temperature T_0 , which is above the solidification temperature T_s . At this time the material occupies a region $r_0 \leq r \leq r_1$, $|\theta| \leq \theta_0$, $|z| \leq z_0$ with respect to appropriate circular cylindrical coordinates. The fibres are assumed to lie in surfaces $r = \text{constant}$, and to be arranged in a manner which is macroscopically consistent with orthotropic symmetry: for example, a balanced angle ply lay-up, comprising alternate unidirectional sheets with fibre directions $\pm\alpha^\circ$ to the z -axis.

For times $t > 0$ the liquid phase deforms in response to cooling from the inner surface $r = r_0$; for simplicity we consider only plane-strain deformations, so that there is no displacement in the z -direction, and the in-plane (r - θ) displacements are independent of z .

Cooling continues until time t_0 , when the temperature on the inner surface reaches the solidification temperature T_s . Subsequently a solidification front moves through the material and the solid thus created deforms, once more in plane-strain. The heat flux associated with elastic deformation is assumed to be negligible, and so the temperature problem can be solved independently. For simplicity we neglect heat loss from the edges $|\theta| = \theta_0$, and so the heat conduction equation reduces to

$$\kappa_h^{(1)} \frac{\partial^2 T}{\partial r^2} + \frac{\kappa_h^{(2)}}{r} \frac{\partial T}{\partial r} = c_h \rho_h \frac{\partial T}{\partial t}, \quad (1)$$

where $\kappa_h^{(1)}$ and $\kappa_h^{(2)}$ are the conductivities in the radial and angular directions, c_h is the specific heat capacity, and ρ_h is the density (the subscript h takes on the values s or l when referring to the solid or liquid regimes respectively). If the interface between the solid and liquid phases is given by $r = a(t)$, and L is the latent heat of solidification, then balance of heat flux across the interface can be expressed by

$$\kappa_l^{(1)} \frac{\partial T(r, t)}{\partial r} \Big|_{r=a(t)^+} - \kappa_s^{(1)} \frac{\partial T(r, t)}{\partial r} \Big|_{r=a(t)^-} = a'(t)L; \quad (2)$$

clearly T is continuous at the interface $r = a(t)$, and moreover $T(a(t), t) = T_s$. Let $\tau(r)$ be the time at which a particle at radius r solidifies, so that

$$\begin{aligned} T(r, \tau(r)) &= T_s, \\ \tau(a(t)) &= t, \\ a(\tau(r)) &= r. \end{aligned}$$

We assume for the remainder of the paper that equations (1) and (2) can be solved with appropriate boundary conditions, so that the temperature function T is known.

The liquid phase is modelled as an "anisotropic thermoelastic liquid". The radial, angular and axial displacements of particles from their initial configuration at time $t = 0$ are denoted by u , v and $w = 0$, and the corresponding non-trivial strains obey the formulas

$$\begin{aligned} e_{rr} &= u_{,r}, \\ e_{\theta\theta} &= (u + v_{,\theta})/r, \\ 2e_{r\theta} &= u_{,\theta}/r + v_{,r} - v/r, \end{aligned} \quad (3)$$

where subscripted commas indicate differentiation with respect to the indicated variables. The remaining strains

$$\begin{aligned} e_{zz} &= w_{,z}, \\ 2e_{\theta z} &= v_{,z} + w_{,\theta}/r, \\ 2e_{rz} &= u_{,z} + w_{,r}, \end{aligned} \quad (4)$$

are all zero by virtue of the plane-strain assumption. The components of the stress tensor are written $\sigma_{r\theta}$ etc.

We assume that the shearing of surfaces $r = \text{constant}$ over one another is hindered only by the viscosity of the liquid matrix, and that the shear traction on these surfaces is negligible during slow cooling. Thus we assume that $\sigma_{r\theta} = 0$, $\sigma_{rz} = 0$, for the liquid motion; since the deformation has plane-strain we also have $\sigma_{\theta z} = 0$. We also assume that the fibres remain solid but flexible in this phase of the material, and that there is insufficient time for any significant percolation of the liquid matrix between the fibres. This gives the liquid phase some of the characteristics of an orthotropic thermoelastic solid; in particular we propose that the relationship between the principal stresses and strains can be characterised by

$$\begin{pmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \end{pmatrix} = \begin{pmatrix} \bar{c}_{11} & \bar{c}_{12} & \bar{c}_{13} \\ \bar{c}_{12} & \bar{c}_{22} & \bar{c}_{23} \\ \bar{c}_{13} & \bar{c}_{23} & \bar{c}_{33} \end{pmatrix} \begin{pmatrix} e_{rr} - \bar{\alpha}_1(T - T_0) \\ e_{\theta\theta} - \bar{\alpha}_2(T - T_0) \\ e_{zz} - \bar{\alpha}_3(T - T_0) \end{pmatrix}, \quad (5)$$

$$\sigma_{r\theta} = 0, \quad \sigma_{rz} = 0, \quad \sigma_{\theta z} = 0,$$

where the \bar{c}_{ij} are elastic constants and the $\bar{\alpha}_j$ are coefficients of expansion in the liquid.

The solid phase is assumed to behave like a conventional orthotropic thermoelastic solid, and the displacements are denoted U , V and W . Once again, plane strain is assumed, so that $W = 0$ and the following strains are all equal to zero.

$$\begin{aligned} \varepsilon_{zz} &= W_{,z}, \\ 2\varepsilon_{\theta z} &= V_{,z} + W_{,\theta}/r, \\ 2\varepsilon_{rz} &= U_{,z} + W_{,r}. \end{aligned} \quad (6)$$

The non-trivial strain-displacement formulas are

$$\begin{aligned} \varepsilon_{rr} &= U_{,r}, \\ \varepsilon_{\theta\theta} &= (U + V_{,\theta})/r, \\ 2\varepsilon_{r\theta} &= U_{,\theta}/r + V_{,r} - V/r. \end{aligned} \quad (7)$$

However, since these displacements are measured from the configuration at time $t = 0$ (when the material is liquid), the constitutive law is not quite straightforward.

In particular, the liquid phase does not accommodate shear stresses $\sigma_{r\theta}$ or σ_{rz} for the reasons stated earlier; thus any such stress component in the solid phase must be proportional to the *increase* in strain *following* solidification: for example we have

$$\sigma_{r\theta} = 2c_{66}(\varepsilon_{r\theta}|_t - \varepsilon_{r\theta}|_{\tau(r)}) \quad (8)$$

(recall that a particle which is initially at radius r solidifies at time $\tau(r)$). The r - z and θ - z components of stress and strain are automatically zero due to the assumption of plane strain.

Turning now to the direct stresses and strains: the linear constitutive law for a conventional orthotropic thermoelastic material maintains that the direct stresses are proportional to the relative strain, as measured from some natural stress-free state, less a term due to thermal expansion. In the present case, this relative strain is simply the absolute strain as measured from the liquid at temperature T_0 , less both the strain associated with thermal contraction of the liquid phase, and the strain associated with the liquid-to-solid phase change (essentially this is equivalent to measuring the strain in the solid from a hypothetical stress-free configuration of solid at temperature T_s). Subtracting the thermal expansion terms $\alpha_j(T - T_s)$ from these relative strains indicates that the direct stresses are proportional to terms such as $\varepsilon_{rr} - (e_1 + \bar{\alpha}_1(T - T_0)) - \alpha_1(T - T_s)$, where e_j are direct strains arising on solidification. The full constitutive law for plane strain deformations of the solid phase is thus

$$\begin{pmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} \begin{pmatrix} \varepsilon_{rr} - (e_1 + \bar{\alpha}_1(T_s - T_0)) - \alpha_1(T - T_s) \\ \varepsilon_{\theta\theta} - (e_2 + \bar{\alpha}_2(T_s - T_0)) - \alpha_2(T - T_s) \\ \varepsilon_{zz} - (e_3 + \bar{\alpha}_3(T_s - T_0)) - \alpha_3(T - T_s) \end{pmatrix}, \quad (9)$$

$$\begin{aligned} \sigma_{\theta z} &= 0, \\ \sigma_{rz} &= 0, \\ \sigma_{r\theta} &= 2c_{66}(\varepsilon_{r\theta}|_t - \varepsilon_{r\theta}|_{\tau(r)}). \end{aligned}$$

The equations of motion are simply that the divergence of the stress field is zero. For the case of plane strain this reduces to

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= 0, \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} &= 0. \end{aligned} \quad (10)$$

(Note that these simplify further in the liquid region since $\sigma_{r\theta} = 0$.)

It is assumed that there are no in-plane tractions on the boundaries $r = r_0, r_1$ and $|\theta| = \theta_0$. In addition, during the exclusively liquid phase of cooling ($0 < t < t_0$) the liquid is contained by the lower surface $r = r_1$, and hence the condition $u(r_1, t) = 0$ must be satisfied for $0 < t < t_0$.

3. Displacement in the liquid

As mentioned earlier, we seek solutions for which the in-plane stresses are identically zero. However, since $\sigma_{r\theta}$ is automatically zero, there is an arbitrary amount

of shear possible in the solutions. To avoid an unresolved non-uniqueness, we assume that solutions are symmetrical about $\theta = 0$. We make two further simplifying assumptions for both the solid and liquid displacements: namely that circular arcs remain circular arcs throughout, and that at each time t , each circular arc $r = \text{const}$, $z = \text{const}$ has undergone a uniform stretch in the θ direction. Our initial justification for these is largely the simplicity of exposition which they deliver, although it turns out that these simplifications do in fact yield an exact solution to the problem.

After substituting from equations (5) a little rearrangement shows that the in-plane stresses are zero provided that

$$\begin{aligned} u_{,r} &= \bar{\mathcal{L}}_1(\bar{\alpha})(T - T_0), \\ u + v_{,\theta} &= r\bar{\mathcal{L}}_2(\bar{\alpha})(T - T_0). \end{aligned} \quad (11)$$

The quantities $\bar{\mathcal{L}}_i(\bar{\alpha})$ are simply constant material parameters which are explained in the Appendix. The simplifying assumptions outlined above have the effect of restricting the arbitrary functions which result from the integration of equations (11). This integration yields

$$\begin{aligned} u(r, t) &= -\bar{\mathcal{L}}_1(\bar{\alpha}) \int_r^{r_1} (T(s, t) - T_0) ds + f(t), \\ v(r, \theta, t) &= \theta [r\bar{\mathcal{L}}_2(\bar{\alpha})(T - T_0) - u(r, t)]. \end{aligned} \quad (12)$$

Since $u(r_1, t)$ is zero up until time t_0 , the arbitrary function $f(\cdot)$ satisfies

$$f(t) = 0 \text{ for } 0 < t < t_0. \quad (13)$$

4. Displacement in the solid

In line with the simplifications mentioned in the previous section we assume that U is independent of θ , and that V is proportional to θ . In this case, $\sigma_{r\theta}$ is identically zero if and only if

$$V(r, \theta, t) = (F(t) - E(r))r\theta,$$

for some functions $E(\cdot)$ and $F(\cdot)$. After substitution using (9) we find that

$$\begin{aligned} \sigma_{rr} &= c_{11}\varepsilon_{rr} + c_{12}\varepsilon_{\theta\theta} - \mathcal{K}_1(e) - \mathcal{K}_1(\bar{\alpha})(T_s - T_0) - \mathcal{K}_1(\alpha)(T - T_s), \\ \sigma_{\theta\theta} &= c_{12}\varepsilon_{rr} + c_{22}\varepsilon_{\theta\theta} - \mathcal{K}_2(e) - \mathcal{K}_2(\bar{\alpha})(T_s - T_0) - \mathcal{K}_2(\alpha)(T - T_s), \end{aligned}$$

and the solution to $\sigma_{rr} = \sigma_{\theta\theta} = 0$ requires

$$\begin{aligned} U_{,r}(r, t) &= \mathcal{L}_1(\alpha)(T - T_s) + \mathcal{L}_1(\bar{\alpha})(T_s - T_0) + \mathcal{L}_1(e), \\ U(r, t)/r &= E(r) - F(t) + \mathcal{L}_2(\alpha)(T - T_s) + \mathcal{L}_2(\bar{\alpha})(T_s - T_0) + \mathcal{L}_2(e). \end{aligned}$$

(The definition of the constants $\mathcal{L}_i(\alpha)$, $\mathcal{K}_i(e)$ etc are given in the Appendix. In particular, the presence of the symbol e in the constant $\mathcal{K}_1(e)$ denotes that the constant may depend on all three components of the triple $e = (e_1, e_2, e_3)$.) Differentiating the second of these of these equations, and substituting into the first yields

$$\mathcal{L}_2(\alpha)rT_{,r}(r, t) - \mathcal{L}_0(\alpha)(T - T_s) = F(t) - (rE(r))' + \mathcal{L}_0(\bar{\alpha})(T_s - T_0) + \mathcal{L}_0(e). \quad (14)$$

By noting that the right-hand side of (14) is of the form $f_1(t) + f_2(r)$, it is straightforward to show that the solutions of the heat conduction equation (1) that are compatible with (14) are given by

$$T(r, t) = T_s + k(\tau(r) - t), \quad (15)$$

where

$$\tau(r) = k_0 - \frac{r^2 c_s \rho_s}{2\kappa_s^{(1)}(1 + \kappa_s)} + k_1 r^{1 - \kappa_s}, \quad (16)$$

$$\kappa_s = \frac{\kappa_s^{(2)}}{\kappa_s^{(1)}},$$

and where k , k_0 and k_1 are constants. In fact this is a very special temperature distribution, since the cooling rate is equal to the constant k everywhere for $t_0 < t < t_1$. It is also the only solution to the heat conduction equation in which the spatial gradient of the temperature field is constant in time. The significance of this remark will be pointed out in Section 7.

Substituting $T(r, t)$ from (15) into (14), equating the parts dependent on t and r , and integrating, gives

$$F(t) = \mathcal{L}_0(\alpha)kt,$$

$$E(r) = \frac{k\mathcal{L}_1(\alpha)}{r} \int_{r_0}^r \tau(s)ds - k\mathcal{L}_2(\alpha)\tau(r) + \mathcal{L}_0(\bar{\alpha})(T_s - T_0) + \mathcal{L}_0(e) + \frac{P}{r},$$

where P is constant. Hence

$$U(r, t) = k\mathcal{L}_1(\alpha) \int_{r_0}^r \tau(s)ds - k\mathcal{L}_1(\alpha)rt + r\mathcal{L}_1(\bar{\alpha})(T_s - T_0) + r\mathcal{L}_1(e) + P,$$

$$V(r, \theta, t) = \theta \left[rk\mathcal{L}_0(\alpha)t - k\mathcal{L}_1(\alpha) \int_{r_0}^r \tau(s)ds + rk\mathcal{L}_2(\alpha)\tau(r) - r\mathcal{L}_0(\bar{\alpha})(T_s - T_0) - r\mathcal{L}_0(e) - P \right]. \quad (17)$$

5. Solution during solidification

All that now remains is to apply continuity of the displacement at the interface between the solid and liquid phases. Equating $u(a(t), t)$ and $U(a(t), t)$ from

equations (12) and (17) gives

$$f(t) = P + k\mathcal{L}_1(\alpha) \int_{r_0}^{a(t)} \tau(s) ds - kta(t)\mathcal{L}_1(\alpha) + \mathcal{L}_1(\bar{\alpha})a(t)(T_s - T_0) \\ + a(t)\mathcal{L}_1(e) + \bar{\mathcal{L}}_1(\bar{\alpha}) \int_{a(t)}^{r_1} (T(s, t) - T_0) ds.$$

In particular, setting $t = t_0$ and $a(t_0) = r_0$, the constant P is determined as

$$P = f(t_0) + kr_0t_0\mathcal{L}_1(\alpha) - \mathcal{L}_1(\bar{\alpha})r_0(T_s - T_0) - r_0\mathcal{L}_1(e) \\ - \bar{\mathcal{L}}_1(\bar{\alpha}) \int_{r_0}^{r_1} (T(s, t_0) - T_0) ds. \quad (18)$$

From (13) it follows that if $f(t)$ is continuous at $t = t_0$, then $f(t_0) = 0$ and P is determined by (18). However it is possible to produce plausible arguments which suggest that $f(t)$ jumps discontinuously at $t = t_0$ (see [1]). To allow this possibility the term $f(t_0)$ is retained in (18) and subsequently.

When $f(t_0)$ is specified there are no remaining unknowns in the problem, and so it is not possible to ensure continuity of the tangential displacements. There are two possible reactions to this result. The first, and most obvious, is that we have constrained our solution too heavily: there is no stress-free solution of the form we chose. The second possibility, which was also discussed in [2], is that it is inappropriate to demand continuity of the tangential displacement at the interface. Indeed, it has already been assumed that in the liquid phase, surfaces $r = \text{const}$ may slip over one another unhindered by shear stress; to allow discontinuity is merely to suggest that this still holds true at the interface itself. It seems likely that this issue can be fully resolved only by a careful dynamic analysis of the material behaviour at the solid-liquid interface. Indeed this comment applies equally well to the question of a discontinuity in f at t_0 , which was discussed in the previous paragraph.

For the remainder of this paper we adopt the viewpoint which permits the discontinuity in the tangential displacement. Thus the displacements are given by

$$U(r, t) = k\mathcal{L}_1(\alpha) \int_{r_0}^r \tau(s) ds - k\mathcal{L}_1(\alpha)(rt - r_0t_0) - (T_0 - T_s)\mathcal{L}_1(\bar{\alpha})(r - r_0) \\ + \mathcal{L}_1(e)(r - r_0) + \bar{\mathcal{L}}_1(\bar{\alpha}) \int_{r_0}^{r_1} (T_0 - T(s, t_0)) ds + f(t_0), \\ V(r, \theta, t) = \theta \left[rk\mathcal{L}_0(\alpha)t + r\mathcal{L}_0(\bar{\alpha})(T_0 - T_s) - r\mathcal{L}_0(e) - k\mathcal{L}_1(\alpha) \int_{r_0}^r \tau(s) ds \quad (19) \right. \\ \left. + rk\mathcal{L}_2(\alpha)\tau(r) - r_0t_0k\mathcal{L}_1(\alpha) - r_0(T_0 - T_s)\mathcal{L}_1(\bar{\alpha}) + r_0\mathcal{L}_1(e) \right. \\ \left. - \bar{\mathcal{L}}_1(\bar{\alpha}) \int_{r_0}^{r_1} (T_0 - T(s, t_0)) ds - f(t_0) \right].$$

All of the quantities in these equations are known and so they represent the exact stress-free solution to our problem during solidification.

In order to interpret this solution more easily, we make some approximations based on a typical cross-ply or balanced angle-ply fibre-polymer composite:

$$\begin{aligned} \alpha_1 \gg \alpha_2, \alpha_3, \quad c_{22} \gg c_{11}, c_{12}, c_{13}, c_{23}, \quad e_1 \gg e_2, e_3, \\ \bar{\alpha}_1 \gg \bar{\alpha}_2, \bar{\alpha}_3, \quad \bar{c}_{22} \gg \bar{c}_{11}, \bar{c}_{12}, \bar{c}_{13}, \bar{c}_{23}. \end{aligned}$$

It follows (see Appendix) that

$$\mathcal{L}_0(x) \approx \mathcal{L}_1(x) \approx x_1, \quad \mathcal{L}_2(x) \approx 0,$$

whenever x is α , $\bar{\alpha}$, or e . With these approximations (19) simplify to

$$\begin{aligned} U(r, t) &= k\alpha_1 \left[\int_{r_0}^r \tau(s) ds - (rt - r_0 t_0) \right] + e_1(r - r_0) + f(t_0) \\ &\quad - \bar{\alpha}_1 \left[(T_0 - T_s)(r - r_0) - \int_{r_0}^{r_1} (T_0 - T(s, t_0)) ds \right], \quad (20) \\ V(r, \theta, t) &= -\theta U(r, t). \end{aligned}$$

Thus in this approximation $\varepsilon_{\theta\theta} = 0$, so that there is no stretch of material curves which lie in the surfaces $r = \text{constant}$ (indeed this is to be expected since in the limit, our approximations are equivalent to thermally and elastically inextensible fibres). Hence the deformation consists of a radial displacement given by $U(r, t)$ and a shear on surfaces $r = \text{constant}$ for which these surfaces do not stretch, but flex to accommodate this radial motion. Furthermore the deformation consists of a part independent of the cooling rate k , which inspection shows is the result of thermal expansion in the fluid state and the expansion at solidification, and a part proportional to k . Clearly the effect of the cooling rate becomes relatively more significant as k increases. Although this has been demonstrated only for the special temperature distribution (15), it may be conjectured that more general temperature profiles will lead to the qualitatively similar conclusion that rapid cooling rates will have a large effect on the amount of distortion during the solidification phase.

6. Solution after solidification

The change in curvature at the end of solidification was given in the previous section. However, the forming process is not fully complete until the component has cooled to room temperature. This last part of the calculation is a straight-forward problem in linear thermoelasticity: it is necessary to simply add the solution of Section 5 at time t_1 to the solution for a solid circular cylindrical sector undergoing a change in temperature from $T(r, t_1)$ to room temperature T_r . A numerical

solution to the problem is acceptable, as is an approximation based on thin shell theory.

However, the basic behaviour of such a solution can be ascertained with minimal effort, by breaking down this temperature change into two hypothetical parts. First *heat* the solid back up to the solidification temperature T_s (or infinitesimally below it, so that it does not melt). The constitutive law is linear, the boundaries are free of traction and imposed displacement, and the change in temperature during this stage is proportional to the constant k : thus the resulting stress and displacement must also be proportional to k .

Now cool the cylinder to room temperature T_r . The stress and deformation introduced during this stage of cooling must be proportional to $T_s - T_r$, by similar reasoning. In fact, there is no stress introduced during this second hypothetical stage of cooling. This is a result of the so-called spring-forward phenomenon of O'Neill, Rogers and Spencer [3], in which it is shown that an orthotropic thermoelastic solid subjected to a uniform change in temperature curls up but does not develop any stress.

Hence for the complete solidification process, starting with liquid at temperature T_0 and ending with solid at room temperature T_r , we find that:

- The magnitude of the change in curvature increases with both the cooling rate, and the difference between the solidification temperature T_s and the final resting temperature T_r .
- The stress induced is proportional to the cooling rate alone.

7. Discussion and Conclusions

The stress and deformation induced by the solidification of a circular cylindrical sector of thermoelastic fibre-reinforced material have been given. Both the liquid and solid phases were seen to be free of in-plane stresses during solidification, although this can only occur in the specific case when the temperature in the solid is given by equations (15, 16). This is the sole solution to the heat conduction equation for which the spatial gradient of the temperature field at each point remains constant throughout solidification.

It appears that each particle of the solid can accommodate the particular temperature gradient which existed at its creation from liquid: as long as that gradient persists, the material may change its curvature in response to a uniform lowering of temperature, but no in-plane stress will be induced. The curvature change at the instant when solidification ceases, increases with with the cooling rate according to equation (19).

The lack of in-plane stress with the above cooling regime is analogous to the so-called "spring-forward" phenomenon of Watson, Spencer and Rogers [3]. That result concerns the deformation of a solid stress-free cylindrical sector in response to changing temperature: as long as the temperature change at each point of the

sector is the same (or equivalently, the spatial gradient does not change), the sector curls up, but does not develop any in-plane stress.

Once the sector has completely solidified and subsequently cooled to room temperature, both the stress and change in curvature increase with the cooling rate; in particular, the stress is proportional to it. In fact, the limiting case of infinitesimally slow cooling coincides precisely with the spring-forward result mentioned above.

These results may not be welcomed by the processing community, since they indicate that the residual stresses induced by solidification can be minimized only by allowing the cooling rate to fall to zero, ie by extremely slow cooling.

8. Appendix

In order to keep the number of constants to a minimum, we have found it convenient to introduce the following "functional constants". For the constant three-vector $x = (x_1, x_2, x_3)$, define $\mathcal{K}(x)$ and $\bar{\mathcal{K}}(x)$ by

$$\mathcal{K}_i(x) = \sum_{j=1}^3 c_{ij} x_j,$$

$$\bar{\mathcal{K}}_i(x) = \sum_{j=1}^3 \bar{c}_{ij} x_j.$$

Now define $\mathcal{L}_i(x)$ and $\bar{\mathcal{L}}_i(x)$ by

$$\mathcal{L}_1(x) = \frac{c_{22}\mathcal{K}_1(x) - c_{12}\mathcal{K}_2(x)}{c_{11}c_{22} - c_{12}^2},$$

$$\mathcal{L}_2(x) = \frac{c_{11}\mathcal{K}_2(x) - c_{12}\mathcal{K}_1(x)}{c_{11}c_{22} - c_{12}^2},$$

$$\bar{\mathcal{L}}_1(x) = \frac{\bar{c}_{22}\bar{\mathcal{K}}_1(x) - \bar{c}_{12}\bar{\mathcal{K}}_2(x)}{\bar{c}_{11}\bar{c}_{22} - \bar{c}_{12}^2},$$

$$\bar{\mathcal{L}}_2(x) = \frac{\bar{c}_{11}\bar{\mathcal{K}}_2(x) - \bar{c}_{12}\bar{\mathcal{K}}_1(x)}{\bar{c}_{11}\bar{c}_{22} - \bar{c}_{12}^2}.$$

and let

$$\mathcal{L}_0(x) = \mathcal{L}_1(x) - \mathcal{L}_2(x),$$

$$\bar{\mathcal{L}}_0(x) = \bar{\mathcal{L}}_1(x) - \bar{\mathcal{L}}_2(x).$$

These further simplify to

$$\mathcal{L}_0(x) = x_1 - x_2 + x_3 \frac{c_{13}(c_{22} + c_{12}) - c_{23}(c_{11} + c_{12})}{c_{11}c_{22} - c_{12}^2},$$

$$\mathcal{L}_1(x) = x_1 + x_3 \frac{c_{22}c_{13} - c_{12}c_{23}}{c_{11}c_{22} - c_{12}^2},$$

$$\mathcal{L}_2(x) = x_2 + x_3 \frac{c_{11}c_{23} - c_{12}c_{13}}{c_{11}c_{22} - c_{12}^2},$$

with equivalent formulas for $\bar{\mathcal{L}}_i(x)$.

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