LONGITUDINAL FLOWS IN ARRAYS OF PARALLEL FIBRES

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Estimating the longitudinal Darcy permeability of an aligned fibre bed, or the steady-state elongational viscosity of a fibre suspension, involves solving a longitudinal flow around a set of cylinders. This is usually done either by a computational effort, or analytically through a model geometry simplified to the extent that the many-body nature of the problem is eliminated (often involving just a single fibre). My present objective is to establish a mobility method for this type of many-body problem, which can generate closed-form analytical results as far as the complexity of the fibre arrangement allows, as well as numerical ones for more complex models.

The governing equations for this are Poisson's equations in two dimensions,

$$\frac{\partial v^2}{\partial x^2} + \frac{\partial v^2}{\partial y^2} = \frac{1}{\eta}\frac{\partial P}{\partial z}, \tag{1}$$

where $v = v(x, y)$ is the longitudinal (z-parallel) fluid velocity, and $\partial P/\partial z$ is the longitudinal pressure gradient. Any particular problem in this category is distinguished by the position and radius of the boundary curve of each fibre, along with an auxiliary condition on each, for example, a prescribed motion or a prescribed resultant force or couple or any combination of those. In permeability problems the flow is driven entirely by the right hand side, whereas in the elongation of a fibre suspension the pressure gradient is zero and the flow is driven by the relative motion of the rods.

Here, a solution $v(\mathbf{x})$ for the 2D velocity field will be constructed by superposition of a finite set of fundamental solutions, $v_{(\alpha)}(\mathbf{x})$, each of which matches a restricted motion and a particular moment of force associated with a circular curve Γ^α in the x, y-plane. The curves may be used to represent actual internal or external solid boundaries, or other no-slip surfaces such as the stress-free surface in a so-called free surface model of permeability.

Each circle Γ^α has a centre $\mathbf{x}^\alpha$, and a radius a^α and is associated with four velocities,

$$\mathbf{U}^\alpha = (u^\alpha, \omega_x^\alpha, \omega_y^\alpha, \overline{v}^\alpha), \tag{2}$$

where u^α is the linear z-velocity of Γ^α, ω_x^α and ω_y^α are angular velocities about the x- and y-axes and $\overline{v}^\alpha$ is the average velocity within Γ^α, i.e.,

$$\overline{v}^\alpha = \frac{1}{\pi\left(a^\alpha\right)^2}\int_{A^\alpha} v dA. \tag{3}$$

Let $v_{(\alpha)}(\mathbf{x})$ be the fluid velocity field generated by the motion of a single boundary Γ^α, and $v^\alpha(\mathbf{x})$ the velocity of Γ^α itself. The *exact* solution constitutes a sum of four fundamental solutions, or Green's functions, to the two-dimensional Poisson's equations, each of which exactly matches one of the four motions:

$$v_{(\alpha)}(\mathbf{x}) = \begin{cases} \frac{1}{2\pi\eta}\begin{pmatrix} (f^\alpha + p^\alpha)\ln\frac{\rho}{|\mathbf{x}-\mathbf{x}^\alpha|} \\ +l_x^\alpha \frac{y-y^\alpha}{|\mathbf{x}-\mathbf{x}^\alpha|^2} - l_y^\alpha \frac{x-x^\alpha}{|\mathbf{x}-\mathbf{x}^\alpha|^2} \end{pmatrix} & \text{if } |\mathbf{x}-\mathbf{x}^\alpha| \geq a^\alpha \\ \frac{1}{2\pi\eta}\begin{pmatrix} (f^\alpha + p^\alpha)\ln\frac{\rho}{a^\alpha} + l_x^\alpha \frac{y-y^\alpha}{(a^\alpha)^2} \\ -l_y^\alpha \frac{x-x^\alpha}{(a^\alpha)^2} + \frac{1}{2}p^\alpha\left(1 - \frac{|\mathbf{x}-\mathbf{x}^\alpha|^2}{(a^\alpha)^2}\right) \end{pmatrix} & \text{if } |\mathbf{x}-\mathbf{x}^\alpha| < a^\alpha \end{cases} \tag{4}$$

The coefficients

$$\mathbf{F}^\alpha = (f^\alpha, l_x^\alpha, l_y^\alpha, p^\alpha) \tag{5}$$

are moments of force associated with Γ^α: f^α is the total force applied on Γ^α, l_x^α and l_y^α are the applied couples and p^α is the total force per unit length due to the pressure gradient applied inside Γ^α:

$$p^\alpha = -\pi\,(a^\alpha)^2\,\frac{\partial P}{\partial z}. \tag{6}$$

The constant arbitrary radius ρ is introduced in order to make the velocity field determinate, and will eventually cancel out. In the outside solution, all the terms are is singular at $\mathbf{x}^\alpha$. Inside Γ^α, the solution consists of a constant velocity (the first term), simple uniform shear (second and third terms) and a parabolic field (the last term). On Γ^α itself the inside and outside solutions match each other exactly.

Now consider a set of N circular curves Γ^α, each centred at $\mathbf{x}^\alpha$ and having a radius a^α. The many-body solution is obtained by adding up the fields due to all the $4N$ forces, couples and pressure gradients considered:

$$v(\mathbf{x}) = \sum_{\alpha=1}^{N} v_{(\alpha)}(\mathbf{x}). \tag{7}$$

This velocity field is defined everywhere. It is smooth and continuous and satisfies Poisson's equations exactly everywhere except on Γ.

It remains to determine the resultant velocities of each Γ^α so as to approximately satisfy a no-slip condition on the cylinder surfaces. To do this we separate $\mathbf{U}^\alpha$ into four contributions; one due to $\mathbf{F}^\alpha$ itself, one due to other Γs that are completely separate from $\mathbf{x}^\alpha$, one due to any Γs enclosing $\mathbf{x}^\alpha$and one due to Γs enclosed by Γ^α:

$$\begin{aligned} \mathbf{U}^\alpha &= \sum_{\beta=1}^{N} \mathbf{U}^\alpha_{(\beta)} \\ &= \mathbf{U}^\alpha_{(\alpha)} + \sum_{\substack{|\mathbf{x}^\beta-\mathbf{x}^\alpha| \\ >a^\beta+a^\alpha}} \mathbf{U}^\alpha_{(\beta)} + \sum_{\substack{|\mathbf{x}^\beta-\mathbf{x}^\alpha| \\ <a^\beta-a^\alpha}} \mathbf{U}^\alpha_{(\beta)} + \sum_{\substack{|\mathbf{x}^\alpha-\mathbf{x}^\beta| \\ <a^\alpha-a^\beta}} \mathbf{U}^\alpha_{(\beta)}. \end{aligned} \tag{8}$$

For the first set, $\mathbf{U}^{\alpha}_{(\alpha)}$, of cylinder velocities there is only one that matches the field $v_{(\alpha)}$ exactly:

$$
\begin{aligned}
u^{\alpha}_{(\alpha)} &= \frac{f^{\alpha}+p^{\alpha}}{2\pi\eta}\ln\frac{\rho}{a^{\alpha}}, && (9)\\
\omega^{\alpha}_{x(\alpha)} &= \frac{l^{\alpha}_{x}}{2\pi\eta}\frac{1}{(a^{\alpha})^{2}}, && (10)\\
\omega^{\alpha}_{y(\alpha)} &= \frac{l^{\alpha}_{y}}{2\pi\eta}\frac{1}{(a^{\alpha})^{2}}, && (11)\\
\overline{v}_{(\alpha)} &= \frac{1}{2\pi\eta}\left((f^{\alpha}+p^{\alpha})\ln\frac{\rho}{a^{\alpha}}+\frac{1}{4}p^{\alpha}a^{\alpha}\right). && (12)
\end{aligned}
$$

The second set of velocities satisfies the no-slip condition only approximately with the remaining fields since these fields are nonlinear on Γ^{α}. The approximation used here is to match them with the linearisation (tangent plane) of the field about the centre, $\mathbf{x}^{\alpha}$, of Γ^{α}:

$$
\begin{aligned}
\sum_{\substack{|\mathbf{x}^{\beta}-\mathbf{x}^{\alpha}|\\ >a^{\beta}+a^{\alpha}}} u^{\alpha}_{(\beta)} &= \sum_{\substack{|\mathbf{x}^{\beta}-\mathbf{x}^{\alpha}|\\ >a^{\beta}+a^{\alpha}}} v_{(\beta)}(\mathbf{x}^{\alpha}) && (13)\\
&= \frac{1}{2\pi\eta}\sum_{\substack{|\mathbf{x}^{\beta}-\mathbf{x}^{\alpha}|\\ >a^{\beta}+a^{\alpha}}}\begin{pmatrix}(f^{\beta}+p^{\beta})\ln\frac{\rho}{|\mathbf{x}-\mathbf{x}^{\beta}|}\\ +l^{\beta}_{x}\frac{y^{\alpha}-y^{\beta}}{|\mathbf{x}^{\alpha}-\mathbf{x}^{\beta}|^{2}}\\ -l^{\beta}_{y}\frac{x^{\alpha}-x^{\beta}}{|\mathbf{x}^{\alpha}-\mathbf{x}^{\beta}|^{2}}\end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{\substack{|\mathbf{x}^{\beta}-\mathbf{x}^{\alpha}|\\ >a^{\beta}+a^{\alpha}}} \omega^{\alpha}_{(\beta)x} &= \sum_{\substack{|\mathbf{x}^{\beta}-\mathbf{x}^{\alpha}|\\ >a^{\beta}+a^{\alpha}}} \frac{\partial}{\partial y}v_{(\beta)}(\mathbf{x}^{\alpha}) && (14)\\
&= \frac{1}{2\pi\eta}\sum_{\substack{|\mathbf{x}^{\beta}-\mathbf{x}^{\alpha}|\\ >a^{\beta}+a^{\alpha}}}\begin{pmatrix}-(f^{\beta}+p^{\beta})\frac{y^{\alpha}-y^{\beta}}{|\mathbf{x}^{\alpha}-\mathbf{x}^{\beta}|^{2}}\\ +l^{\beta}_{x}\frac{(x^{\alpha}-x^{\beta})^{2}-(y^{\alpha}-y^{\beta})^{2}}{|\mathbf{x}^{\alpha}-\mathbf{x}^{\beta}|^{4}}\\ +2l^{\beta}_{y}\frac{(x^{\alpha}-x^{\beta})(y^{\alpha}-y^{\beta})}{|\mathbf{x}^{\alpha}-\mathbf{x}^{\beta}|^{4}}\end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{\substack{|\mathbf{x}^{\beta}-\mathbf{x}^{\alpha}|\\ >a^{\beta}+a^{\alpha}}} \omega^{\alpha}_{(\beta)y} &= -\sum_{\substack{|\mathbf{x}^{\beta}-\mathbf{x}^{\alpha}|\\ >a^{\beta}+a^{\alpha}}} \frac{\partial}{\partial x}v_{(\beta)}(\mathbf{x}^{\alpha}) && (15)\\
&= \frac{1}{2\pi\eta}\sum_{\substack{|\mathbf{x}^{\beta}-\mathbf{x}^{\alpha}|\\ >a^{\beta}+a^{\alpha}}}\begin{pmatrix}(f^{\beta}+p^{\beta})\frac{x^{\alpha}-x^{\beta}}{|\mathbf{x}^{\alpha}-\mathbf{x}^{\beta}|^{2}}\\ +2l^{\beta}_{x}\frac{(x^{\alpha}-x^{\beta})(y^{\alpha}-y^{\beta})}{|\mathbf{x}^{\alpha}-\mathbf{x}^{\beta}|^{4}}\\ +l^{\beta}_{y}\frac{(y^{\alpha}-y^{\beta})^{2}-(x^{\alpha}-x^{\beta})^{2}}{|\mathbf{x}^{\alpha}-\mathbf{x}^{\beta}|^{4}}\end{pmatrix},
\end{aligned}
$$

$$\sum_{\substack{|\mathbf{x}^\beta-\mathbf{x}^\alpha| \\ >a^\beta+a^\alpha}} \overline{v}^\alpha_{(\beta)} = \sum_{\substack{|\mathbf{x}^\beta-\mathbf{x}^\alpha| \\ >a^\beta+a^\alpha}} v_{(\beta)}\left(\mathbf{x}^\alpha\right) \tag{16}$$

$$= \frac{1}{2\pi\eta} \sum_{\substack{|\mathbf{x}^\beta-\mathbf{x}^\alpha| \\ >a^\beta+a^\alpha}} \begin{pmatrix} \left(f^\beta+p^\beta\right)\ln\frac{\rho}{|\mathbf{x}-\mathbf{x}^\beta|} \\ +l^\beta_x \frac{y^\alpha-y^\beta}{|\mathbf{x}^\alpha-\mathbf{x}^\beta|^2} \\ -l^\beta_y \frac{x^\alpha-x^\beta}{|\mathbf{x}^\alpha-\mathbf{x}^\beta|^2} \end{pmatrix}.$$

The third set, produced by sources Γ^β completely enclosing Γ^α, is obtained from the internal part of the field (4). Although this may not be immediately evident, the inside field of Equation 4 is linear on any circle within Γ^β, and hence satisfies the no-slip condition on Γ^α exactly:

$$\sum_{\substack{|\mathbf{x}^\beta-\mathbf{x}^\alpha| \\ <a^\beta-a^\alpha}} u^\alpha_{(\beta)} = \frac{1}{2\pi\eta} \sum_{\substack{|\mathbf{x}^\beta-\mathbf{x}^\alpha| \\ <a^\beta-a^\alpha}} \begin{pmatrix} \left(f^\beta+p^\beta\right)\ln\frac{\rho}{a^\beta} \\ +l^\beta_x \frac{y^\alpha-y^\beta}{\left(a^\beta\right)^2} - l^\beta_y \frac{x^\alpha-x^\beta}{\left(a^\beta\right)^2} \\ +\frac{1}{2}p^\beta\left(1-\frac{|\mathbf{x}^\alpha-\mathbf{x}^\beta|^2}{\left(a^\beta\right)^2}-\left(\frac{a^\alpha}{a^\beta}\right)^2\right) \end{pmatrix}, \tag{17}$$

$$\sum_{\substack{|\mathbf{x}^\beta-\mathbf{x}^\alpha| \\ <a^\beta-a^\alpha}} \omega^\alpha_{x(\beta)} = \frac{1}{2\pi\eta} \sum_{\substack{|\mathbf{x}^\beta-\mathbf{x}^\alpha| \\ <a^\beta-a^\alpha}} \left(l^\beta_x \frac{1}{\left(a^\beta\right)^2} - p^\beta \frac{y^\alpha-y^\beta}{\left(a^\beta\right)^2}\right), \tag{18}$$

$$\sum_{\substack{|\mathbf{x}^\beta-\mathbf{x}^\alpha| \\ <a^\beta-a^\alpha}} \omega^\alpha_{y(\beta)} = \frac{1}{2\pi\eta} \sum_{\substack{|\mathbf{x}^\beta-\mathbf{x}^\alpha| \\ <a^\beta-a^\alpha}} \left(l^\beta_y \frac{1}{\left(a^\beta\right)^2} + p^\beta \frac{x^\alpha-x^\beta}{\left(a^\beta\right)^2}\right), \tag{19}$$

$$\sum_{\substack{|\mathbf{x}^\beta-\mathbf{x}^\alpha| \\ <a^\beta-a^\alpha}} \overline{v}^\alpha_{(\beta)} = \sum_{\substack{|\mathbf{x}^\beta-\mathbf{x}^\alpha| \\ <a^\beta-a^\alpha}} \frac{1}{\pi\left(a^\alpha\right)^2} \int_{A^\alpha} v_{(\beta)} dA \tag{20}$$

$$= \frac{1}{2\pi\eta} \sum_{\substack{|\mathbf{x}^\beta-\mathbf{x}^\alpha| \\ <a^\beta-a^\alpha}} \begin{pmatrix} \left(f^\beta+p^\beta\right)\ln\frac{\rho}{a^\beta} \\ +l^\beta_x \frac{y^\alpha-y^\beta}{\left(a^\beta\right)^2} - l^\beta_y \frac{x^\alpha-x^\beta}{\left(a^\beta\right)^2} \\ +\frac{1}{2}p^\beta\left(1-\frac{|\mathbf{x}^\alpha-\mathbf{x}^\beta|^2}{\left(a^\beta\right)^2}-\frac{1}{2}\left(\frac{a^\alpha}{a^\beta}\right)^2\right) \end{pmatrix}.$$

The fourth set, due to a source Γ^β completely enclosed by Γ^α may be omitted here due to symmetry.

Substituting (9)–(20) into (8), we can write the latter equations in the form:

$$
\begin{Bmatrix} u^1 \\ \vdots \\ u^N \\ \omega_x^1 \\ \vdots \\ \omega_x^N \\ \omega_y^1 \\ \vdots \\ \omega_y^N \\ \overline{v}^1 \\ \vdots \\ \overline{v}^N \end{Bmatrix} = \frac{1}{2\pi\eta} \begin{bmatrix} \begin{bmatrix} A^{11} & A^{12} & \cdots & A^{1N} \\ A^{12} & A^{22} & & A^{2N} \\ \vdots & & & \vdots \\ A^{1N} & A^{2N} & \cdots & A^{NN} \end{bmatrix} & [D] & [E] & [H] \\ \left[\widetilde{D}\right] & [B] & [F] & [I] \\ \left[\widetilde{E}\right] & \left[\widetilde{F}\right] & [C] & [J] \\ \left[\widetilde{H}\right] & \left[\widetilde{I}\right] & \left[\widetilde{J}\right] & [G] \end{bmatrix} \begin{Bmatrix} f^1 \\ \vdots \\ f^N \\ l_x^1 \\ \vdots \\ l_x^N \\ l_y^1 \\ \vdots \\ l_y^N \\ p^1 \\ \vdots \\ p^N \end{Bmatrix},
$$

where tilde denotes transpose, and the mobilities are as follows:

	$\beta = \alpha$	β, α separated	β enclosing α
$A^{\alpha\beta} = A^{\beta\alpha}$	$\ln \frac{\rho}{a^\alpha}$	$\ln \frac{\rho}{\lvert \mathbf{x}^\alpha - \mathbf{x}^\beta \rvert}$	$\ln \frac{\rho}{a^\beta}$
$B^{\alpha\beta} = B^{\beta\alpha}$	$(a^\alpha)^{-2}$	$\frac{(x^\alpha - x^\beta)^2 - (y^\alpha - y^\beta)^2}{\lvert \mathbf{x}^\alpha - \mathbf{x}^\beta \rvert^4}$	$(a^\alpha)^{-2}$
$C^{\alpha\beta} = C^{\beta\alpha}$	$(a^\alpha)^{-2}$	$\frac{(y^\alpha - y^\beta)^2 - (x^\alpha - x^\beta)^2}{\lvert \mathbf{x}^\alpha - \mathbf{x}^\beta \rvert^4}$	$(a^\alpha)^{-2}$
$D^{\alpha\beta}$	0	$\frac{y^\alpha - y^\beta}{\lvert \mathbf{x}^p - \mathbf{x}^\beta \rvert^2}$	$\frac{y^\alpha - y^\beta}{(a^\beta)^2}$
$D^{\beta\alpha}$	0	$-\frac{y^\alpha - y^\beta}{\lvert \mathbf{x}^p - \mathbf{x}^\beta \rvert^2}$	0
$E^{\alpha\beta}$	0	$-\frac{x^\alpha - x^\beta}{\lvert \mathbf{x}^\alpha - \mathbf{x}^\beta \rvert^2}$	$-\frac{x^\alpha - x^\beta}{(a^\beta)^2}$
$E^{\beta\alpha}$	0	$\frac{x^\alpha - x^\beta}{\lvert \mathbf{x}^\alpha - \mathbf{x}^\beta \rvert^2}$	0
$F^{\alpha\beta} = F^{\beta\alpha}$	0	$2\frac{(x^\alpha - x^\beta)(y^\alpha - y^\beta)}{\lvert \mathbf{x}^\alpha - \mathbf{x}^\beta \rvert^4}$	0
$G^{\alpha\beta} = G^{\beta\alpha}$	$\ln \frac{\rho}{a^\alpha} + \frac{1}{4}$	$\ln \frac{\rho}{\lvert \mathbf{x}^\alpha - \mathbf{x}^\beta \rvert}$	$\ln \frac{\rho}{a^\beta} + \frac{1}{2}\left(1 - \left(\frac{\lvert \mathbf{x}^\alpha - \mathbf{x}^\beta \rvert}{a^\beta}\right)^2 - \frac{1}{2}\left(\frac{a^\alpha}{a^\beta}\right)^2\right)$
$H^{\alpha\beta}$	$\ln \frac{\rho}{a^\alpha}$	$\ln \frac{\rho}{\lvert \mathbf{x}^\alpha - \mathbf{x}^\beta \rvert}$	$\ln \frac{\rho}{a^\beta} + \frac{1}{2}\left(1 - \left(\frac{\lvert \mathbf{x}^\alpha - \mathbf{x}^\beta \rvert}{a^\beta}\right)^2 - \left(\frac{a^\alpha}{a^\beta}\right)^2\right)$
$H^{\beta\alpha}$	$\ln \frac{\rho}{a^\alpha}$	$\ln \frac{\rho}{\lvert \mathbf{x}^\alpha - \mathbf{x}^\beta \rvert}$	$\ln \frac{\rho}{a^\beta}$
$I^{\alpha\beta}$	0	$-\frac{y^\alpha - y^\beta}{\lvert \mathbf{x}^p - \mathbf{x}^\beta \rvert^2}$	$-\frac{y^\alpha - y^\beta}{(a^\beta)^2}$
$I^{\beta\alpha}$	0	$\frac{y^\alpha - y^\beta}{\lvert \mathbf{x}^p - \mathbf{x}^\beta \rvert^2}$	0
$J^{\alpha\beta}$	0	$\frac{x^\alpha - x^\beta}{\lvert \mathbf{x}^\alpha - \mathbf{x}^\beta \rvert^2}$	$\frac{x^\alpha - x^\beta}{(a^\beta)^2}$
$J^{\beta\alpha}$	0	$-\frac{x^\alpha - x^\beta}{\lvert \mathbf{x}^\alpha - \mathbf{x}^\beta \rvert^2}$	0

These equations should be solved along with a condition of self-equilibrium,

$$\sum_{\alpha=1}^{N} f^{\alpha} + \sum_{\alpha=1}^{N} p^{\alpha} = 0, \tag{22}$$

$$\sum_{\alpha=1}^{N} l_x^{\alpha} = 0, \tag{23}$$

$$\sum_{\alpha=1}^{N} l_y^{\alpha} = 0, \tag{24}$$

to ensure that the velocity field is steady and vanishes at infinity.

The mobility formulation is more general than earlier methods. The mobility matrix can be solved analytically, for a small number of fibres in regular arrays, or numerically for general arrangements including non-uniform cylinder diameters. The numerical analysis of general arrays should answer questions such as the impact of fibre arrangement in suspensions and fibre beds, and how many cylinders need to be included in order to model an infinite array.

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Nomenclature list

η	=	Newtonian viscosity of suspending liquid
$\mathbf{x}, x, y, z$	=	co-ordinates
N	=	number of cylinders, $\alpha, \beta = 1, \cdots, N$
a^{α}	=	radius of cylinder α
Γ^{α}	=	intersection curve of cylinder α with the x, y-plane
$\mathbf{F}^{\alpha}$	=	generalised force vector $(f^{\alpha}, l_x^{\alpha}, l_x^{\alpha})$ for cylinder α
$\mathbf{U}^{\alpha}$	=	generalised velocity vector $(u^{\alpha}, \omega_x^{\alpha}, \omega_y^{\alpha})$ for cylinder α
$\mathbf{U}_{(\beta)}^{\alpha}$	=	generalised velocity vector for cylinder α due to $\mathbf{F}^{\beta}$
f^{α}	=	force per unit length on fluid by cylinder α
$l_x^{\alpha}, l_y^{\alpha}$	=	couple per unit length on fluid by cylinder α
p^{α}	=	force per unit length of cylinder α due to pressure gradient
u^{α}	=	longitudinal velocity of cylinder α
$\omega_x^{\alpha}, \omega_y^{\alpha}$	=	angular velocities of cylinder α
$\overline{v}^{\alpha}$	=	average fluid velocity within cylinder α
$A^{\alpha\beta}, \cdots, J^{\alpha\beta}$	=	mobilities
$v(\mathbf{x})$	=	total velocity field
$v_{(\alpha)}(\mathbf{x})$	=	velocity field due to $\mathbf{F}^{\alpha}$